

A Landau-Ginzburg model for Lagrangian Grassmannians, Langlands duality and relations in quantum cohomology

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1 Introduction

For a complex simple, simply connected algebraic group G and parabolic subgroup P , the homogeneous space G/P has a Landau-Ginzburg model defined by the second author [Rie08], which is a regular function on an affine subvariety of the Langlands dual group and is shown in [Rie08] to recover the Peterson variety presentation [Pet97] of the quantum cohomology of G/P . In the case of type A Grassmannians R. Marsh and the second author [MR12] reformulated this Landau-Ginzburg model as a rational function on a Langlands dual Grassmannian, and used this formulation to prove a version of the mirror symmetry conjecture about flat sections of the A-model connection stated in [BCFKvS00].

In this paper we formulate an LG-model (\check{X}, W_t) for G/P in the case of a Lagrangian Grassmannian in the spirit of the mirrors of the type A Grassmannians, and prove that it is isomorphic to the LG-model from [Rie08]. This LG model has some very interesting features, which are not visible in the type A case, to do with the non-triviality of Langlands duality. We also formulate an explicit conjecture relating our superpotential with the quantum differential equations of $LG(m)$. Finally, our expression for W_t also leads us to conjecture new formulas in the quantum Schubert calculus of $LG(m)$.

To give an idea of our result, which is very explicit, we give the first two interesting examples here. Note that the Schubert basis of $H^*(LG(m))$ is indexed by strict partitions λ fitting in an $m \times m$ box and can be identified with coordinates p_λ on the Grassmannian $OG^{\text{co}}(m+1, 2m+1)$ of $(m+1)$ -dimensional co-isotropic subspaces of \mathbb{C}^{2m+1} endowed with a non-degenerate quadratic form. Note that $OG^{\text{co}}(m+1, 2m+1)$ is canonically isomorphic to the maximal orthogonal Grassmannian $OG(m, 2n+1)$. Moreover, it is related to X by Langlands duality. The goal of this paper is to give an explicit description of a Landau-Ginzburg model for $LG(m)$ as a rational

function on $\text{OG}^{\text{co}}(m+1, 2m+1)$. As an example, for $LG(2)$ our Landau-Ginzburg model is the rational function on $\text{OG}^{\text{co}}(3, 5)$ given by

$$W_t = \frac{p_{\square}}{p_{\emptyset}} + \frac{p_{\square\square}^2}{p_{\square}p_{\square\square} - p_{\emptyset}p_{\square\square\square}} + e^t \frac{p_{\square}}{p_{\square\square}}.$$

For $LG(3)$ we obtain the rational function on $\text{OG}^{\text{co}}(4, 7)$,

$$W_t = \frac{p_{\square}}{p_{\emptyset}} + \frac{p_{\square\square\square\square} - p_{\emptyset}p_{\square\square\square\square}}{p_{\square}p_{\square\square\square} - p_{\emptyset}p_{\square\square\square\square}} + \frac{p_{\square\square\square\square}p_{\square\square\square\square} - p_{\square\square\square\square}p_{\square\square\square\square}}{p_{\square\square}p_{\square\square\square} - p_{\square\square}p_{\square\square\square}} + e^t \frac{p_{\square\square}}{p_{\square\square\square}}.$$

We generalise these formulas and prove that they agree with the superpotential from [Rie08] after suitable identifications.

Notice how the above formulas have 3, 4 summands, these numbers being the index of $X = LG(2)$, $LG(3)$, respectively. Indeed this comes from the fact that in all of the cases W_t represents the anti-canonical class of X in a natural sense (in the Jacobi ring for example), and each summand represents a hyperplane class. On the other hand, because W_t wants to be regular in the complement of an anti-canonical divisor, the degrees of the denominators in W_t should add up to the index of \check{X} . That is, in the above two cases to 4 and 6, these being the index of $\text{OG}^{\text{co}}(3, 5)$ and $\text{OG}^{\text{co}}(4, 7)$, respectively. This is exactly what is achieved by the quadratic terms in the $LG(m)$ cases, with $1 + 2 + 1 = 4$, and $1 + 2 + 2 + 1 = 6$ (and so forth, in our general formula).

For usual Grassmannians X and \check{X} are isomorphic so have the same index. Therefore numerators and denominators in W_t are allowed to be sections of $\mathcal{O}(1)$. This leads to the formulas in [MR12] looking more compact.

2 Background

In [Rie08], the second author gave a Lie-theoretic construction of a Landau-Ginzburg model of any complete homogeneous space X of a simple complex algebraic group. The LG-model (\check{X}°, W) is set in the world of the Langlands dual group.

2.1 Notation

Let X be a complete homogeneous space for a simple complex algebraic group. For the purposes of this paper we will denote the group acting on X by G^\vee and assume that G^\vee is simply connected, and we will denote its Langlands dual group by G , which is therefore an adjoint group. For G^\vee we may fix Chevalley generators $(e_i^\vee)_{1 \leq i \leq m}$ and $(f_i^\vee)_{1 \leq i \leq m}$ and correspondingly Borel subgroups $B_+^\vee = T^\vee U_+^\vee$ and $B_-^\vee = T^\vee U_-^\vee$. We may assume that $X = G^\vee / P^\vee$ for a parabolic subgroup P^\vee which contains B_+^\vee . The parabolic

P^\vee is determined by a choice of subset of the $(f_i^\vee)_{1 \leq i \leq m}$. This set also determines a parabolic subgroup P of G , where we also have the analogous Borel subgroups $B_+ = TU_+$ and $B_- = TU_-$ and Chevalley generators $(e_i)_{1 \leq i \leq m}$ and $(f_i)_{1 \leq i \leq m}$. Let $\Pi = \{\alpha_i \mid i \in I\}$ denote the set of simple roots. The set of all roots is $R = R^+ \sqcup R^-$, where R^+ is the subset of positive roots and R^- the subset of negative roots.

Denote by W the Weyl group of G (canonically identified with the Weyl group of G^\vee), and let W_P be the Weyl group of the parabolic subgroup P . Let T^{W_P} be the W_P -fixed sub-torus. If α is a positive root, we denote by $s_\alpha \in W$ the associated reflection. Let R_P^+ be the set of all positive roots α such that $s_\alpha \in W_P$, Π_P be the set of simple roots in R_P^+ , and $\Pi^P = \Pi \setminus \Pi_P$. When $\alpha = \alpha_i$ is a simple root, we set $s_i := s_{\alpha_i}$. Moreover, we denote the length of $w \in W$ by $\ell(w)$. It is equal to the minimum number of simple reflections whose product is w . We also let w_0 and w_P , be the longest elements in W and W_P , respectively, and define W^P to be the set of minimal length coset representatives for W/W_P . The minimal length coset representative for w_0 is denoted by w^P , so that $w_0 = w^P w_P$. Let \dot{w} denote a representative of $w \in W$ in G .

Using the exponential map we may think of U_+ and U_- as being embedded in the completed universal enveloping algebra $\hat{\mathcal{U}}_+$, respectively $\hat{\mathcal{U}}_-$. Accordingly $e_i^*(u)$ will denote the coefficient of e_i in $u \in U_+$ after this embedding, and analogously for f_i^* and $\bar{u} \in U_-$.

2.2 Quantum cohomology of G/P

The quantum cohomology ring of a smooth complex projective variety X is a deformation of its cohomology ring. While the cohomology ring of X encodes the way its subvarieties intersect each other, the quantum cohomology ring encodes the way they are connected by rational curves. The structure constants of the (small) quantum cohomology ring are called Gromov-Witten invariants. When $X = G/P$ is homogeneous, Gromov-Witten invariants count the number of rational curves of given degree intersecting three given Schubert varieties of X .

The quantum cohomology rings of a full flag variety was first described by Givental and B. Kim [GK95, Kim99], who related it to a degenerate leaf of the Toda lattice of the Langlands dual group. Soon after, Dale Peterson came up with a new point of view in which all of the quantum cohomology rings of complete homogeneous spaces for one group are encoded in terms of strata of one remarkable subvariety of the Langlands dual full flag variety. This so-called *Peterson variety* \mathbb{Y} is defined as follows. In our conventions Peterson's variety \mathbb{Y} encoding the quantum cohomology rings of G^\vee -homogeneous spaces is a subvariety of G/B_- . Denote by \mathfrak{n}_- the Lie algebra of U_- , and by $[\mathfrak{n}_-, \mathfrak{n}_-]$ its commutator subalgebra. The annihilator in \mathfrak{g}^* of a subspace \mathfrak{l} of \mathfrak{g} is denoted by \mathfrak{l}^\perp . Consider the coadjoint action of

G on \mathfrak{g}^* and the ‘principal nilpotent’ element $F = \sum e_i^*$ in \mathfrak{g}^* . Then

$$\mathbb{Y} := \{gB_- \mid g^{-1} \cdot F \in [\mathfrak{n}_-, \mathfrak{n}_-]^\perp\}.$$

First note that this variety has an open stratum $Y_B = \mathbb{Y} \cap (B_+B_-/B_-)$ which is isomorphic to the degenerate leaf of the Toda lattice for G via the map $Y_B \hookrightarrow \mathfrak{g}^*$ defined by $u_+B_- \mapsto u_+^{-1} \cdot F$. By Peterson’s theory, the quantum cohomology rings for all other G^\vee/P^\vee are described by the coordinate rings of the smaller strata $Y_P = \mathbb{Y} \cap (B_+\dot{w}_PB_-/B_-)$, where we take the intersection in the possibly non-reduced sense.

Theorem 2.1 (Peterson). *The quantum cohomology of G^\vee/P^\vee is isomorphic to the coordinate ring $\mathbb{C}[Y_P]$ of the stratum Y_P of the Peterson variety \mathbb{Y} .*

In [FW04], Fulton and Woodward proved a quantum Chevalley formula for $X = G/P$, i.e. a formula giving the product of an arbitrary Schubert class by any Schubert class associated to a Schubert divisor. Here we state this formula, which we will refer to in Section 4. Note that for $P = B$, the formula is a result of Peterson [Pet97].

If s_i is a simple reflection, we denote by $\Gamma_i \in H_2(X, \mathbb{Z})$ the associated dimension 1 Schubert cycle, and we define, for $\alpha \in R^+ \setminus R_P^+$:

$$d(\alpha) := \sum_{i=1}^m \alpha^\vee(\omega_i) \Gamma_i.$$

Now set $q^{d(\alpha)} := \prod_{i=1}^m q_i^{\alpha^\vee(\omega_i)}$, where q_i is the quantum parameter associated to Γ_i . Finally, for $\alpha \in R^+ \setminus R_P^+$, we define $n_\alpha := \int_{\Gamma_\alpha} c_1(TX)$, where $\Gamma_\alpha \in H_2(X, \mathbb{Z})$ is the dimension 1 cycle associated to the reflection s_α (it is a linear combination of the Γ_i).

Theorem 2.2 ([FW04]). *For $1 \leq i \leq m$ and $w \in W^P$ we have*

$$\sigma_{s_i} \star \sigma_w = \sum_{\alpha} \alpha^\vee(\omega_i) \sigma_{ws_\alpha} + \sum_{\alpha} q^{d(\alpha)} \alpha^\vee(\omega_i) \sigma_{ws_\alpha},$$

where the first sum is over roots $\alpha \in R^+ \setminus R_P^+$ such that $l(ws_\alpha) = l(w) + 1$, and the second sum over roots $\alpha \in R^+ \setminus R_P^+$ such that $l(ws_\alpha) = l(w) + 1 - n_\alpha$.

2.3 The Lie-theoretic LG model construction

We recall how the mirror Landau-Ginzburg models are defined in [Rie08]. Let us fix a parabolic P . We consider the open Richardson variety $\mathcal{R} := R_{w_P, w_0} \subset G/B_-$, namely

$$\mathcal{R} := R_{w_P, w_0} = (B_+\dot{w}_PB_- \cap B_-\dot{w}_0B_-)/B_-.$$

Instead of the whole stratum Y_P of the Peterson variety the LG-model is related to the open dense subset $Y_P^* := \mathbb{Y} \cap \mathcal{R}$, whose coordinate ring in Peterson's theory encodes the quantum cohomology ring $qH^*(G^\vee/P^\vee)$ with quantum parameters made invertible. We note that in this setting if $g = u_1 d \dot{w}_P \bar{u}_2 = b_- \dot{w}_0$ represents an element $gB_- \in \mathcal{R}$ lying in Y_P^* , then the values of the functions on Y_P^* corresponding to the quantum parameters are just the values $\alpha_j(d)$ for the simple roots $\alpha_j \in \Pi^P$. Indeed, fixing $d \in T^{W_P}$ determines a finite subscheme of $Y_P^* = \mathbb{Y} \cap \mathcal{R}$ which we denote by $Y_{P,d}^* = Y_P^* \times_{T^{W_P}} \{d\}$ and for which the non-reduced coordinate ring $\mathbb{C}[Y_{P,d}^*]$ becomes identified with the quantum cohomology ring of G^\vee/P^\vee with quantum parameters fixed to the values $\alpha_j(d)$ in Peterson's theory.

Now let us define

$$Z = Z_{G^\vee/P^\vee} := B_- \dot{w}_0 \cap U_+ T^{W_P} \dot{w}_P U_-.$$

There is an isomorphism

$$\begin{aligned} Z & \rightarrow \mathcal{R} \times T^{W_P}, \\ g = u_1 d \dot{w}_P \bar{u}_2 = b_- \dot{w}_0 & \mapsto (gB_-, d). \end{aligned}$$

Observe that $gB_- = b_- \dot{w}_0 B_- = u_1 \dot{w}_P B_-$. Note that our conventions differ from [Rie08] in that we have translated the original definition of the variety Z by \dot{w}_0 . The mirror superpotential to $X = G^\vee/P^\vee$ is now defined to be the regular function $\mathcal{F} : Z \rightarrow \mathbb{C}$ defined by

$$\mathcal{F}(u_1 d \dot{w}_P \bar{u}_2) = \sum_{i=1}^m e_i^*(u_1) + \sum_{i=1}^m f_i^*(\bar{u}_2). \quad (1)$$

Although u_1 and \bar{u}_2 are not uniquely determined for $g \in Z$, the function \mathcal{F} is well-defined, as was shown in [Rie08]. Actually, there is another small difference with [Rie08], in that in [Rie08] the group on the mirror side is assumed adjoint, whereas here we have assumed G to be simply connected. However we could have carried out the above definitions for $G/\text{Center}(G)$, and in the following it will not matter.

The superpotential \mathcal{F} may also be interpreted as a family of functions $\mathcal{F}_h : \mathcal{R} \rightarrow \mathbb{C}$ depending holomorphically on a parameter $h \in \mathfrak{h}^{W_P}$, by setting

$$\mathcal{F}_h(u_1 \dot{w}_P B_-) = \sum_{i=1}^m e_i^*(u_1) + \sum_{i=1}^m f_i^*(\bar{u}_2) \quad (2)$$

where $u_1 \in U_+$ and $u_1 \dot{w}_P B_- \in \mathcal{R}$, and where $\bar{u}_2 \in U_-$ is related to u_1 by $u_1 e^h \dot{w}_P \bar{u}_2 \in Z$. Equivalently the relationship between u_1 and \bar{u}_2 can be expressed as

$$\bar{u}_2 \cdot B_+ = e^{-h} \dot{w}_P^{-1} u_1^{-1} \cdot B_-.$$

where $g \cdot B$ denotes the conjugation action of $g \in G$ on a Borel subgroup B .

The main result in [Rie08] describes the critical point scheme of \mathcal{F}_h as subscheme of \mathcal{R} lying inside the Peterson variety. We denote by Y_{P,e^h}^* the (non-reduced) fiber over e^h of the Peterson variety, namely

$$Y_{P,e^h}^* = Y_P^* \times_{TW_P} \{e^h\}.$$

Theorem 2.3 ([Rie08]). *The critical point scheme of \mathcal{F}_h agrees with Y_{P,e^h}^* .*

Putting this together with Peterson's presentation this result can be interpreted as follows. Suppose $h \in \mathfrak{h}^{W_P}$ represents a Kaehler class $[\omega_h]$ under the identification $\mathfrak{h}^{W_P} = H^2(G^\vee/P^\vee)$.

Corollary 2.4. *The Jacobi ring $\mathbb{C}[Z_h]/(\partial\mathcal{F}_h)$ of $\mathcal{F}_h : Z_h \rightarrow \mathbb{C}$ is isomorphic to the quantum cohomology ring of the Kaehler manifold $(G^\vee/P^\vee, [\omega_h])$ in its presentation due to Dale Peterson [Pet97].*

In [MR12], R. Marsh and the second author gave an expression of the Landau-Ginzburg model of the Grassmannian in terms of Plücker coordinates and then described the A-model connection. Here we will express the Landau-Ginzburg model of the Lagrangian Grassmannian in terms of *generalized Plücker coordinates*, i.e the coordinates of its minimal embedding $\mathrm{OG}^{\mathrm{co}}(m+1, 2m+1) \hookrightarrow \mathbb{P}(V_{\mathrm{Spin}})$.

3 The Lagrangian Grassmannian and its LG model

Let $G^\vee = \mathrm{PSp}_{2m}(\mathbb{C})$, the adjoint group of type C_m , with Dynkin diagram

$$\begin{array}{ccccccc} \circ & \text{---} & \circ & \text{---} & \cdots & \text{---} & \circ & \Leftarrow & \circ \\ 1 & & 2 & & & & & & m \end{array}.$$

Let $P^\vee := P_{\omega_m^\vee}$ be the parabolic subgroup associated to the m -th fundamental weight ω_m^\vee of G^\vee . The quotient G^\vee/P^\vee is the homogeneous space called the *Lagrangian Grassmannian*, which parametrizes Lagrangian subspaces in \mathbb{C}^{2m} . It is also denoted by $X = \mathrm{LG}(m)$ and will play the role of the A-model for us.

Now the Langlands dual group G is the simply connected group of type B_m , namely the spin group $\mathrm{Spin}_{2m+1}(\mathbb{C})$,

$$\begin{array}{ccccccc} \circ & \text{---} & \circ & \text{---} & \cdots & \text{---} & \circ & \Rightarrow & \circ \\ 1 & & 2 & & & & & & m \end{array}.$$

The parabolic subgroup of $\mathrm{Spin}_{2m+1}(\mathbb{C})$ associated to the m -th fundamental weight is denoted $P = P_{\omega_m}$. In this (B-model) setting we consider the quotient from the left $\check{X} := P \backslash G$. This quotient may be interpreted as the co-isotropic Grassmannian $\mathrm{OG}^{\mathrm{co}}(m+1, 2m+1)$ in a vector space of row vectors. We consider it in its minimal embedding, namely the homogeneous

space $\check{X} := P \backslash G$ is embedded in $\mathbb{P}(V_{\omega_m}^*)$ as right G -orbit of the highest weight vector w_0^* . We will express the mirror Landau-Ginzburg model to $LG(m)$ as a rational function on the orthogonal Grassmannian \check{X} in the homogeneous coordinates of this embedding.

Remark. Note that the Lagrangian Grassmannian $X = LG(m)$ is a cominiscule homogeneous space of type C_m , and therefore its cohomology appears in geometric Satake correspondence [Lus83, MV07, Gin97] as

$$H^*(LG(m)) = IH^*(\overline{Gr}_G^{\omega_m}) = V_{\omega_m}^{\text{Spin}_{2m+1}}.$$

In other words it is canonically identified with the unique miniscule representation, the spin representation $V_{\omega_m}^{\text{Spin}_{2m+1}}$ also denoted V_{Spin} , of the Langlands dual group, $G = \text{Spin}(2m+1)$. Therefore, essentially tautologically, $\mathbb{P}(V_{\text{Spin}}^*)$ has homogeneous coordinates given by the Schubert basis of $H^*(LG(m))$.

3.1 Notations and conventions

Let v_1, \dots, v_{2m+1} be the standard basis of $V = \mathbb{C}^{2m+1}$, and fix the symmetric non-degenerate bilinear form

$$\langle v_i, v_{2m+2-j} \rangle = 2\Phi(v_i, v_{2m+2-j}) = (-1)^{m+1-i} \delta_{i,j}.$$

We may also use the notation $\bar{v}_j = v_{2m+2-j}$ (with decreasing j) for the basis elements v_{m+2}, \dots, v_{2m+1} and set $\epsilon(i) := (-1)^{m+1-i}$ so that $\Phi(v_i, \bar{v}_i) = \epsilon(i)$. The subspace of V spanned by the first m basis vectors v_1, \dots, v_m is maximal isotropic and denoted by W .

We let $G = \text{Spin}(V) = \text{Spin}(V, \Phi)$, which is the universal covering group of $SO(V, \Phi)$. The Lie algebra of $G = \text{Spin}(V)$ is therefore $\mathfrak{so}(V) = \mathfrak{so}(V, \Phi)$ which we view as lying in $\mathfrak{gl}(V)$. We have explicit Chevalley generators e_i, f_i given by

$$\begin{aligned} e_i &= E_{i,i+1} + E_{2m+1-i,2m+2-i} \quad \text{for } i = 1, \dots, m-1, \\ e_m &= \sqrt{2}E_{m,m+1} + \sqrt{2}E_{m+1,m+2}, \\ f_i &= e_i^T \quad \text{for } i = 1, \dots, m. \end{aligned}$$

Here $E_{i,j}$ is the $(2m+1) \times (2m+1)$ -matrix with (i,j) -entry 1 and all other entries 0. We also define the corresponding group homomorphisms $x_i : \mathbb{C} \rightarrow G$ and $y_i : \mathbb{C} \rightarrow G$, namely $x_i(a) := \exp(ae_i)$ and $y_i(a) := \exp(af_i)$.

Next we introduce notations for the Clifford algebra $\text{Cl}(V)$ and the Spin representation V_{Spin} , see also [Var04] whose conventions we follow for the most part. The Clifford algebra $\text{Cl}(V)$ is the algebra quotient of the tensor algebra $T(V)$ by the ideal generated by the expressions

$$v \otimes v' + v' \otimes v - 2\Phi(v, v').$$

So it is the algebra with generators v_{m+1} and v_i, \bar{v}_i for $i = 1, \dots, m$, with relations

$$v_i \bar{v}_i + \bar{v}_i v_i = \epsilon(i), \quad v_{m+1}^2 = \frac{1}{2},$$

and where all other generators anti-commute. The Clifford algebra is $\mathbb{Z}/2\mathbb{Z}$ -graded, as the relations are in even degrees only, and the even part of $\text{Cl}(V)$ is denoted by $\text{Cl}^+(V)$.

Since $\text{Spin}(V)$ acts on V , it acts on $\bigwedge^\bullet V$, and because it preserves the bilinear form Φ , it also acts on $\text{Cl}(V)$. The anti-symmetrization map

$$\begin{aligned} \bigwedge^k V &\rightarrow \text{Cl}(V) \\ v_{i_1} \wedge \dots \wedge v_{i_k} &\mapsto \frac{1}{k!} \left(\sum_{\sigma \in S_k} v_{i_{\sigma(1)}} v_{i_{\sigma(2)}} \cdots v_{i_{\sigma(k)}} \right). \end{aligned}$$

is an embedding of representations, and we will usually identify elements of $\bigwedge^k V$ with their images, as we are mainly interested in the algebra structure of the Clifford algebra. The representation $\bigwedge^2 V$ is isomorphic to the adjoint representation. Moreover the image of $\bigwedge^2 V$ in $\text{Cl}(V)$ is indeed a Lie algebra under the commutator Lie bracket of $\text{Cl}(V)$, and it is isomorphic to $\mathfrak{so}(V)$ as such. In particular our generators e_i, f_i can be identified with elements of $\bigwedge^2 V$ and their images in $\text{Cl}(V)$. Under this identification they are given by

$$\begin{aligned} e_i &= \epsilon(i+1) v_i \wedge \bar{v}_{i+1} = \epsilon(i+1) v_i \bar{v}_{i+1} \text{ for } i = 1, \dots, m-1 \\ e_m &= \sqrt{2} v_m \wedge v_{m+1} = \sqrt{2} v_m v_{m+1}, \\ f_i &= \epsilon(i) v_{i+1} \wedge \bar{v}_i = \epsilon(i) v_{i+1} \bar{v}_i \text{ for } i = 1, \dots, m-1, \\ f_m &= \sqrt{2} \bar{v}_m \wedge v_{m+1} = \sqrt{2} \bar{v}_m v_{m+1}. \end{aligned}$$

Putting all of the anti-symmetrization maps together gives an isomorphism of $\mathfrak{so}(V)$ -modules

$$\bigwedge^\bullet V \longrightarrow \text{Cl}(V).$$

Moreover the even wedge powers map to the even part $\text{Cl}^+(V)$ of the Clifford algebra and odd ones to the odd part, $\text{Cl}^-(V)$. Therefore we have two isomorphisms of $\mathfrak{so}(V)$ -modules

$$\alpha_+ : \bigwedge^{\text{even}} V \longrightarrow \text{Cl}^+(V), \quad (3)$$

$$\alpha_- : \bigwedge^{\text{odd}} V \longrightarrow \text{Cl}^-(V). \quad (4)$$

The Spin representation, as a vector space, is $V_{\text{Spin}} = \bigwedge^\bullet W$. Its standard basis elements are the elements $w_I := v_{i_1} \wedge \dots \wedge v_{i_k}$ with $i_1 < i_2 < \dots < i_k$, where $I = \{i_1, \dots, i_k\}$ is any subset of $\{1, \dots, m\}$. We sometimes write $[v_{i_1} \wedge \dots \wedge v_{i_k}]$ instead of $v_{i_1} \wedge \dots \wedge v_{i_k}$ when we mean the element of V_{Spin} . Note that if $I = \emptyset$ then $w_\emptyset = [1]$.

The subsets I are also in one-to-one correspondence with strict partitions λ contained in an $m \times m$ square, by sending the empty set to the empty partition, and

$$I = \{i_1, \dots, i_k\} \mapsto \lambda = (m+1-i_1, m+1-i_2, \dots, m+1-i_k).$$

In this correspondence the k -row partitions correspond to the basis elements in the k -th graded component, $\bigwedge^k W$, of V_{Spin} . We may denote w_I also by w_λ . If λ is a strict partition contained in an $m \times m$ rectangle, then we denote by $|\lambda|$ the sum of all its parts and by $\text{PD}(\lambda)$ the Poincaré dual partition.

The Spin representation of $\mathfrak{so}(V)$ extends to a representation of the Clifford algebra, which can be defined on generators by

$$v_i \cdot w_I = v_i \wedge w_I, \quad v_{m+1} \cdot w_I = \frac{(-1)^{|I|}}{\sqrt{2}} w_I, \quad \bar{v}_j \cdot w_I = i_{\bar{v}_j}(w_I),$$

where $i_{\bar{v}_i}$ is the insertion operator on $\bigwedge^\bullet W$, for \bar{v}_i identified with the linear form $2\Phi(\bar{v}_i, \cdot)$ on W .

We recall the important fact that the even subalgebra $\text{Cl}^+(V)$ of the Clifford algebra is isomorphic to $\text{End}(V_{\text{Spin}})$ via the action just defined. Combined with the map (3) we obtain an isomorphism of $\mathfrak{so}(V)$ -modules

$$\kappa_+ : \bigwedge^{\text{even}} V \longrightarrow \text{End}(V_{\text{Spin}}). \quad (5)$$

Moreover there is also an isomorphism of $\mathfrak{so}(V)$ -modules,

$$\kappa_- : \bigwedge^{\text{odd}} V \longrightarrow \text{End}(V_{\text{Spin}}) \quad (6)$$

given by antisymmetrization, $\alpha_- : \bigwedge^{\text{odd}} \rightarrow \text{Cl}^-(V)$ followed by the action of $\text{Cl}^-(V)$ on V_{Spin} .

The standard basis $\{w_I\}$ of V_{Spin} defined above is also precisely the integral weight basis obtained by successively applying generators e_i to the lowest weight vector $w_\emptyset = [1]$, and it agrees with the MV -basis of V_{Spin} , which in this case is one and the same as the Schubert basis of $H^*(LG(m))$. We will use the notation σ_λ for the Schubert basis element naturally identified with w_λ .

The generalized Plücker coordinates on our $\text{OG}^{\text{co}}(m+1, 2m+1) = P \backslash G$ are the sections of $\mathcal{O}(1)$ in the embedding $P \backslash G \hookrightarrow \mathbb{P}(V_{\text{Spin}}^*)$ which are given by the basis elements w_λ of V_{Spin} described above. Explicitly, we define

$$p_\lambda(g) := \langle w_\emptyset^* \cdot g, w_\lambda \rangle = w_\emptyset^*(g \cdot w_\lambda),$$

where w_\emptyset^* is the dual basis vector to w_\emptyset , which is therefore a highest weight vector of $V_{\omega_m}^*$, and where w_λ is as above. We may think of an element $Pg \in \text{OG}^{\text{co}}(m+1, V^*) = P \backslash G$ as specified by its homogeneous coordinates

$(p_{\lambda_1}(g) : p_{\lambda_2}(g) : \dots : p_{\lambda_{2m}}(g))$, where $\lambda_1, \dots, \lambda_{2m}$ are the strict partitions in $m \times m$ in some ordering.

To summarize, associated to strict partitions $\lambda \subset m \times m$, or equivalently subsets I of $\{1, \dots, m\}$, we have elements

$$\sigma_\lambda \in H^*(LG(m)), \quad w_\lambda \in V_{\text{Spin}}, \quad \text{and} \quad p_\lambda \in \Gamma[\mathcal{O}_{\text{OG}^{\text{co}}(m+1, V^*)}(1)],$$

all canonically identified. We may also denote them by σ_I, w_I and p_I , respectively.

For a later section we will also require an explicit isomorphism $V \cong V^*$. Since V has on it a quadratic form, we have that $V \cong V^*$ by $v \mapsto \langle v, \rangle$ and V^* has basis $v_1^*, \dots, v_{m+1}^*, v_{m+2}^*, \dots, v_{2m+1}^*$. Under the isomorphism with V this basis corresponds to

$$\begin{aligned} v_1^* &= \epsilon(1)\bar{v}_1 & v_{2m+1}^* &= \bar{v}_1^* = \epsilon(1)v_1 \\ v_2^* &= \epsilon(2)\bar{v}_2 & v_{2m}^* &= \bar{v}_2^* = \epsilon(2)v_2 \\ &\vdots & &\vdots \\ v_m^* &= -\bar{v}_m & v_{m+2}^* &= \bar{v}_m^* = -v_m \\ & & v_{m+1}^* &= v_{m+1}. \end{aligned}$$

3.2 Definition of W_t

We will now explain our formula for $W_t : \text{OG}^{\text{co}}(m+1, V^*) \rightarrow \mathbb{C}$ in terms of the coordinates p_λ . Here are some particular partitions which will play an important role. Let $\rho_l := (l, l-1, \dots, 2, 1)$ be the length l staircase partition and let $\mu_l := (m, m-1, \dots, m+1-l)$ be the maximal strict partition with l lines contained in an $m \times m$ rectangle. For ρ_l with $l < m$ there is a unique strict partition obtained by adding a single box to the Young diagram. It is obtained by adding one box to the first line, and we denote it by $\rho_{l,+}$. If J is any subset of $\{1, \dots, l\}$, we denote by ρ_l^J the partition obtained after removing for every $j \in J$ the j -th line from the Young diagram of ρ_l (and similarly for $\rho_{l,+}^J$). On the other hand we denote by μ_l^J the partition obtained by adding for each $j \in J$ a row of $l+1-j$ boxes to the bottom of μ_l . Similarly, $\mu_{l,+}^J$ is obtained by adding for each $j \in J$ a row of $l+1-j+\delta_{j,1}$ boxes to the bottom of μ_l . If the resulting Young diagram does not give a strict partition, then we set $\mu_l^J = 0$, respectively $\mu_{l,+}^J = 0$. Finally, set $s(J) := \sum_{j \in J} j$ for any subset J of $\{1, \dots, m\}$.

Using the above notations, we define $W_t : \text{OG}^{\text{co}}(m+1, V^*) \rightarrow \mathbb{C}$ by

$$W_t := \frac{p_{\rho_{0,+}}}{p_{\rho_0}} + \sum_{l=1}^{m-1} \frac{\sum_{J \subset \{1, \dots, l\}} (-1)^{s(J)} p_{\rho_{l,+}^J} p_{\mu_{l,+}^J}}{\sum_{J \subset \{1, \dots, l\}} (-1)^{s(J)} p_{\rho_l^J} p_{\mu_l^J}} + e^t \frac{p_{\rho_{m-1}}}{p_{\rho_m}}. \quad (7)$$

This is a rational function on $\check{X} = \text{OG}^{\text{co}}(m+1, V^*)$. Inside \check{X} the denominators in W_t give rise to divisors

$$D_0 := \{p_\emptyset = 0\}, \quad D_m := \{p_{\rho_m} = 0\}$$

and

$$D_l := \left\{ \sum_{J \subset \{1, \dots, l\}} (-1)^{s(J)} p_{\rho_l^J} p_{\mu_l^J} = 0 \right\}, \quad \text{where } l = 1, \dots, m-1.$$

Then

$$D := D_0 + D_1 + \dots + D_{m-1} + D_m$$

is an anti-canonical divisor. Indeed, the index of $\check{X} = \text{OG}^{\text{co}}(m+1, V^*)$ is $2m$. We define $\check{X}^\circ := \check{X} \setminus D$. The restriction of our rational function W_t to \check{X}° is regular, and is again denoted W_t .

We would like to compare $W_t : \check{X}^\circ \rightarrow \mathbb{C}$ with the known super-potential of $X = LG(m)$ defined as a special case of (2). Explicitly recall that $LG(m) = G^\vee / P^\vee$ for $G^\vee = PSp(2m)$ with P^\vee the parabolic corresponding to the m -th node of the Dynkin diagram C_m . The function \mathcal{F}_h for $h \in \mathfrak{h}^{W_P}$ is therefore defined on the open Richardson variety $\mathcal{R} = B_+ w_P B_- \cap B_- \dot{w}_0 B_- / B_-$ inside the full flag variety of $G = \text{Spin}(V)$, where P is the parabolic corresponding to the m -th node of B_m . So we would like to relate our variety $\check{X} = P \backslash G = \text{OG}^{\text{co}}(m+1, V^*)$, or rather its open part \check{X}° , with this open Richardson variety. The parameter t in W_t and the $h \in \mathfrak{h}^{W_P}$ appearing in \mathcal{F}_h should be thought of as equivalent, by the relation $h = t\omega_m^\vee$.

For fixed parameter t we define the following maps

$$\begin{array}{ccccc} \text{OG}^{\text{co}}(m+1, V^*) = P \backslash G & \xleftarrow{\Psi_L} & B_- \dot{w}_0 \cap U_+ e^{t\omega_m^\vee} \dot{w}_P \dot{U}_- & \xrightarrow{\Psi_R} & \mathcal{R}, \\ & & Pg \longleftarrow & g & \longrightarrow gB_- . \end{array}$$

given by taking left and right cosets, respectively. Note that $g = b_- \dot{w}_0$ in our previous notation and factorizes as

$$g = u_1 e^{t\omega_m^\vee} \dot{w}_P \bar{u}_2,$$

Moreover Ψ_R is an isomorphism, so we have $\Psi := \Psi_L \circ \Psi_R^{-1} : \mathcal{R} \rightarrow \text{OG}^{\text{co}}(m+1, V^*)$. Our main goal here is to prove the theorem.

Theorem 3.1. *Let $X = LG(m)$ and $t \in \mathbb{C}$. The rational function W_t on $\text{OG}^{\text{co}}(m+1, V^*)$ defined in (7) pulls back under $\Psi = \Psi_L \circ \Psi_R^{-1}$ to the Landau-Ginzburg model \mathcal{F}_h from Theorem 2.4, where h and t are related by $h = t\omega_m^\vee$.*

The theorem implies that Ψ maps \mathcal{R} to \check{X}° . We also expect the following Claim which we aim to prove in a future version of this paper.

Claim 1. *Ψ defines an isomorphism from \mathcal{R} to \check{X}° .*

Let $h = t\omega_m^\vee$ as in the theorem, and define $Z_h := B_- \dot{w}_0 \cap U_+ e^h \dot{w}_P \dot{w}_0^{-1} U_-$. The super-potential \mathcal{F}_h pulls back under $Z_h \rightarrow G/B_-$ to $\tilde{\mathcal{F}}_h : Z_h \rightarrow \mathbb{C}$ where

$$\tilde{\mathcal{F}}_h(u_1 e^h \dot{w}_P \bar{u}_2) = \sum_{i=1}^m e_i^*(u_1) + \sum_{i=1}^m f_i^*(\bar{u}_2).$$

To prove the theorem we need to show that W_t pulls back to $\tilde{\mathcal{F}}_h$ under $Z_h \dot{w}_0 \xrightarrow{\Psi_L} P \backslash G = \text{OG}^{\text{co}}(m+1, 2m+1)$. We will do this in two steps.

We consider two related projective embeddings of $\tilde{X} = \text{OG}^{\text{co}}(m+1, V^*)$, the standard one corresponding to $\bigwedge^{m+1} V^* = V_{2\omega_m}^*$, and the minimal one corresponding to the (right) representation $V_{\text{Spin}}^* = V_{\omega_m}^*$ of $G = \text{Spin}(V)$ composed with its Veronese embedding. So

$$\begin{aligned} \pi_1 : P \backslash G &\hookrightarrow \mathbb{P}(\bigwedge^{m+1} V^*), \\ Pg &\mapsto \langle v_{m+1}^* \wedge v_{m+2}^* \wedge \cdots \wedge v_{2m+1}^* \cdot g \rangle, \\ \pi_2 : P \backslash G &\hookrightarrow \mathbb{P}(\text{Sym}^2(V_{\text{Spin}}^*)), \\ Pg &\mapsto \langle (w_\emptyset^* \cdot w_\emptyset^*) \cdot g \rangle. \end{aligned}$$

The interesting numerators and denominators in W_t are made up of sections in $\Gamma[\mathcal{O}_{\mathbb{P}(\text{Sym}^2(V_{\text{Spin}}^*))}(1)] = \text{Sym}^2(V_{\text{Spin}})$. However the pullback of $\tilde{\mathcal{F}}_h$ to \tilde{X} is not easy to reformulate directly in those terms. It can be more easily expressed in terms of sections in $\Gamma[\mathcal{O}_{\mathbb{P}(\bigwedge^{m+1} V^*)}(1)] = \bigwedge^{m+1} V$, which correspond to $(m+1) \times (m+1)$ -minors. The two embeddings are however related by an embedding of projective spaces coming from the inclusion of representations

$$\bigwedge^{m+1} V^* \hookrightarrow \text{Sym}^2(V_{\text{Spin}}^*),$$

Therefore dually we have a surjection of representations

$$\text{Sym}^2(V_{\text{Spin}}) \twoheadrightarrow \bigwedge^{m+1} V, \tag{8}$$

which is the restriction map $\Gamma[\mathcal{O}_{\mathbb{P}(\text{Sym}^2(V_{\text{Spin}}^*))}(1)] \rightarrow \Gamma[\mathcal{O}_{\mathbb{P}(\bigwedge^{m+1} V^*)}(1)]$.

The first step of the proof of the theorem is to express $\tilde{\mathcal{F}}_h$ in terms of $(m+1) \times (m+1)$ -minors. The second step involves the explicit construction of the above restriction map, and showing that the degree 2 numerators and denominators in our formula for W_t go to the minors appearing in the first step. From this we go on to deduce that $\psi_L^* W_t$ agrees with $\tilde{\mathcal{F}}_h$.

3.3 A formula for $\tilde{\mathcal{F}}_h$ in terms of minors

Definition 3.1. If $g \in \text{Spin}(V)$ we consider it as acting from the right on $\bigwedge^n V^*$ and from the left on $\bigwedge^n V$ for any $n = 1, \dots, 2m+1$. The bases $\{v_i^*\}$ and $\{v_i\}$ give rise to bases of $\bigwedge^n V^*$ and $\bigwedge^n V$, and we use the following notation for the matrix coefficients (minors of g acting in the representation V). Let $I = \{i_1 < \dots < i_r\}$ be a set indexing rows, and $J = \{j_1 < \dots < j_r\}$ a set indexing columns, then

$$\Delta_J^I(g) := \langle v_{i_1}^* \wedge \cdots \wedge v_{i_r}^* \cdot g, v_{j_1} \wedge \cdots \wedge v_{j_r} \rangle.$$

We begin by arguing that \bar{u}_2 appearing in $u_1 e^h \dot{w}_P \bar{u}_2 \in Z_h$ can be assumed to lie in $U_- \cap B_+ (\dot{w}^P)^{-1} B_+$. This is because we have two birational maps

$$\begin{aligned} \Psi_1 : U_- \cap B_+ (\dot{w}^P)^{-1} B_+ &\rightarrow P \backslash G : & \bar{u}_2 &\mapsto P \bar{u}_2, \\ \Psi_2 : B_- \cap U^+ e^h \dot{w}^P U_- &\rightarrow P \backslash G : & b_- = u_1 e^h \dot{w}_P \bar{u}_2 &\mapsto P b_-, \end{aligned}$$

which compose to give $\Psi_1^{-1} \circ \Psi_2 : b_- \mapsto \bar{u}_2$. This gives a birational map

$$\Psi_1^{-1} \circ \Psi_2 : Z_h \rightarrow U_- \cap B_+ (\dot{w}^P)^{-1} B_+.$$

Now a generic element \bar{u}_2 in $U_- \cap B_+ (\dot{w}^P)^{-1} B_+$ can be assumed to have a particular factorisation. Let $N := \binom{m+1}{2}$. The smallest representative w^P in W of $[w_0] \in W/W_P$ has the following reduced expression :

$$w^P = (s_m)(s_{m-1}s_m) \dots (s_1 s_2 \dots s_m) = s_{i_1} \dots s_{i_N},$$

It follows that as a generic element of $U_- \cap B_+ (\dot{w}^P)^{-1} B_+$, the element \bar{u}_2 can be assumed to be written as:

$$(y_m(a_{m,m}) y_{m-1}(a_{m-1,m}) \dots y_1(a_{1,m})) \dots (y_m(a_{m,2}) y_{m-1}(a_{m-1,2})) y_m(a_{m,1}).$$

where $a_{i,j} \neq 0$, or equivalently as

$$\bar{u}_2 = y_m(b_N) \dots y_2(b_{N-m+2}) y_1(b_{N-m+1}) \dots y_m(b_3) y_{m-1}(b_2) y_m(b_1). \quad (9)$$

with nonzero b_i . Note that the k -th factor here is $y_{i_{N-k+1}}(b_{N-k+1})$.

We may think of the Plücker coordinate p_λ as a function on G . Then we have the following standard expression for the p_λ on factorized elements.

Lemma 3.2. *Fix λ a strict partition in an $m \times m$ square, and $w \in W^P$ the corresponding Weyl group element. Note that the length $\ell(w)$ equals $|\lambda|$. Then if \bar{u}_2 is of the form (9) we have*

$$p_\lambda(\bar{u}_2) = \sum_J b_{j_1} \dots b_{j_m}.$$

where the sum is over subsets $J = \{j_1 < j_2 < \dots < j_m\}$ of $\{1, \dots, N\}$ for which $s_{i_{j_1}} \dots s_{i_{j_m}}$ is a reduced expression of w .

Proof. Recall that by definition $p_\lambda(\bar{u}_2) = \langle w_\emptyset^* \cdot \bar{u}_2, w_\lambda \rangle = w_\emptyset^*(\bar{u}_2 \cdot w_\lambda)$ and $w_\lambda = e_{i_{j_1}} \dots e_{i_{j_m}} \cdot w_\emptyset$ if $w = s_{i_{j_1}} \dots s_{i_{j_m}}$ is a reduced expression. So in an expansion for \bar{u}_2 the coefficients of $f_{i_{j_m}} \dots f_{i_{j_1}}$ will contribute a summand of $b_{j_1} \dots b_{j_m}$ to $p_\lambda(\bar{u}_2)$. \square

Proposition 3.3. *If u_1 and \bar{u}_2 are as above then we have the following identities*

$$f_m^*(\bar{u}_2) = \frac{p_{\rho_0,+}(\bar{u}_2)}{p_{\rho_0}(\bar{u}_2)}, \quad (10)$$

$$e_i^*(u_1) = 0 \text{ for all } 1 \leq i \leq m-1, \quad (11)$$

$$e_m^*(u_1) = e^t \frac{p_{\rho_{m-1}}(\bar{u}_2)}{p_{\rho_m}(\bar{u}_2)}, \quad (12)$$

where $\rho_0 = \emptyset$ and $\rho_{0,+} = \square$.

Proof. For (10) notice that in fact $p_{\emptyset}(\bar{u}_2) = 1$ and

$$p_{\square}(\bar{u}_2) = \langle w_{\emptyset}^* \cdot \bar{u}_2, u_{\square} \rangle = w_{\emptyset}^*(\bar{u}_2 \cdot u_{\square}).$$

Then (10) is apparent since $f_m \cdot u_{\square} = w_{\emptyset}$. In fact (10) does not depend on the special form of u_1 and \bar{u}_2 . The equations (11) and (12) are consequences of the Lemmas A.2 and A.3, respectively, as well as the Lemma 3.2. \square

Proposition 3.4.

$$f_j^*(\bar{u}_2) = \frac{\Delta_{j,j+2,\dots,j+m+1}^{m+1,\dots,2m+1}(\bar{u}_2)}{\Delta_{j+1,\dots,j+m+1}^{m+1,\dots,2m+1}(\bar{u}_2)} \text{ for all } 1 \leq j \leq m-1 \quad (13)$$

Proof. The result is a consequence of the vanishing of the following minor of \bar{u}_2 :

$$\Delta_{j,j+1,\dots,j+m+1}^{j+1,m+1,\dots,2m+1}(\bar{u}_2),$$

which is equal to

$$\langle v_{j+1}^* \wedge v_{m+1}^* \cdots \wedge v_{2m+1}^* \cdot g, v_j \wedge v_{j+1} \cdots \wedge v_{j+m+1} \rangle.$$

Define an element in the enveloping algebra

$$\underline{e} := \left(e_m^{(a_{1,m})} e_{m-1}^{(a_{1,m-1})} \cdots e_1^{(a_{1,1})} \right) \cdots \left(e_m^{(a_{m-1,m})} e_{m-1}^{(a_{m-1,m-1})} \right) e_m^{(a_{m,m})},$$

where $a_{i,j} \in \{0, 1, 2\}$ if $j = m$ and $a_{i,j} \in \{0, 1\}$ otherwise. Here $e_i^{(a)} = \frac{1}{a!} e_i^a$. Due to the shape of \bar{u}_2 , the minor is zero if for any such \underline{e} , $v_{j+1}^* \wedge v_{m+1}^* \cdots \wedge v_{2m+1}^* \cdot \underline{e}$ has zero $v_j^* \wedge v_{j+1}^* \cdots \wedge v_{j+m+1}^*$ -component. Assume by contradiction that there exists an \bar{e} such that this component is nonzero.

First suppose $j = m-1$. Then since $v_m^* \wedge v_{m+1}^* \cdots \wedge v_{2m+1}^* \cdot e_m = 0$, the exponent $a_{1,m}$ in \underline{e} has to be zero. Now the v_{2m+1}^* has to be moved to v_{2m}^* , which means that v_m^* needs to be moved *before* to v_{m-1}^* by an e_{m-1} . Since only one e_1 appears in the expression of \underline{e} , it means that $a_{1,m-1} = 1$. Hence $v_m^* \wedge v_{m+1}^* \cdots \wedge v_{2m+1}^* \cdot \underline{e}$ is equal to

$$v_{m-1}^* \wedge v_{m+1}^* \wedge \cdots \wedge v_{2m+1}^* \cdot (e_{m-2}^{a_{1,m-2}} \cdots e_1^{a_{1,1}}) \cdots (e_m^{a_{m-1,m}} e_{m-1}^{a_{m-1,m-1}}) e_m^{a_{m,m}}.$$

Since $v_{m-1}^* \wedge v_{m+1}^* \wedge \cdots \wedge v_{2m+1}^* \cdot e_i = 0$ for all $1 \leq i \leq m-2$, it follows that $a_{1,m-2} = \cdots = a_{1,2} = a_{1,1} = 0$, which means that the v_{2m+1}^* can never be moved to v_{2m}^* . Hence there exists no \underline{e} such that $v_{j+1}^* \wedge v_{m+1}^* \cdots \wedge v_{2m+1}^* \cdot \underline{e}$ has nonzero $v_j^* \wedge v_{j+1}^* \cdots \wedge v_{j+m+1}^*$ -component.

Now suppose $j < m-1$. v_{2m+1}^* has to be moved to v_{2m}^* by the only e_1 in the expression of \underline{e} , hence $a_{1,1} = 1$. But $v_{m+1}^*, \dots, v_{2m}^*$ need to be moved before, hence $a_{1,i} = 1$ for $1 \leq i \leq m-1$ and $a_{1,m} = 2$. It follows that $v_{j+1}^* \wedge v_{m+1}^* \cdots \wedge v_{2m+1}^* \cdot \underline{e}$ is equal to

$$(v_{j+1}^* \wedge v_m^* \cdots \wedge v_{2m}^* + v_1^* \wedge v_m^* \cdots \wedge v_{m-j}^* \wedge v_{m+2-j}^* \wedge \cdots \wedge v_{2m+1}^*) \cdot \underline{e}',$$

where

$$\underline{e}' := (e_m^{a_{2,m}} e_{m-1}^{a_{2,m-1}} \cdots e_2^{a_{2,2}}) \cdots (e_m^{a_{m-1,m}} e_{m-1}^{a_{m-1,m-1}}) e_m^{a_{m,m}}.$$

Then

$$v_1^* \wedge v_m^* \cdots \wedge v_{m-j}^* \wedge v_{m+2-j}^* \wedge \cdots \wedge v_{2m+1}^* \cdot \underline{e}'$$

has clearly no non-zero $v_j^* \wedge v_{j+1}^* \cdots \wedge v_{j+m+1}^*$ -component, hence we focus on $v_{j+1}^* \wedge v_m^* \cdots \wedge v_{2m}^* \cdot \underline{e}'$.

If $j = m-2$, then v_{2m}^* has to be moved to v_{2m-1}^* by the only e_2 in \underline{e}' . Hence $a_{2,2} = 1$. But $v_{m-1}^* \wedge v_m^* \cdots \wedge v_{2m}^* \cdot e_m = 0$, which means that $a_{2,m} = 0$. It follows that v_{m+1}^* cannot be moved to v_m^* before having to move the v_{2m}^* , and hence that a suitable \underline{e} does not exist.

Finally if $j \leq m-3$, then $v_{j+1}^* \wedge v_m^* \cdots \wedge v_{2m}^* \cdot e_i = 0$ for all $j+1 \leq i \leq m$, hence $a_{2,j+1} = \cdots = a_{2,m} = 0$. It follows that the v_{m+1-j}^* cannot be moved before the v_{2m}^* has to be by the only remaining e_2 in \underline{e}' . This concludes the proof of the minor vanishing.

To prove the proposition, we only need to expand this vanishing minor with respect to the $(j+1)$ -st row. Indeed, due to \bar{u}_2 being lower triangular, this row has only two non-zero entries : 1 on the $(j+1)$ -st column and $f_j^*(\bar{u}_2)$ on the j -th column. \square

3.4 The Clifford Algebra and homogeneous coordinates

3.4.1 Setting

In this section we study the surjection of representations from (8), that is

$$\pi : \text{Sym}^2(V_{\text{Spin}}) \rightarrow \bigwedge^{m+1} V,$$

which is also interpreted as the restriction map of homogeneous coordinates

$$\Gamma[\mathcal{O}_{\mathbb{P}(\text{Sym}^2(V_{\text{Spin}}^*))}(1)] \rightarrow \Gamma[\mathcal{O}_{\mathbb{P}(\bigwedge^{m+1} V^*)}(1)].$$

Of course in representation-theoretic terms the map π exists just because $\bigwedge^{m+1} V$ is irreducible with highest weight $2\omega_m$ and this highest weight also occurs in $\text{Sym}^2(V_{\text{Spin}})$ with multiplicity one. But in order to compute with this map we will need to use a more intrinsic construction. We first note the following auxiliary lemma, whose proof is straightforward.

Lemma 3.5. *The isomorphism*

$$\begin{aligned} \delta : V_{\text{Spin}} &\rightarrow V_{\text{Spin}}^* \\ v_\lambda &\mapsto (-1)^{|\lambda|} v_{\text{PD}(\lambda)} \end{aligned}$$

is $\mathfrak{so}(V)$ -equivariant.

For the construction of the map π first we define an equivariant embedding

$$\iota_{V_{\text{Spin}}} : \text{Sym}^2(V_{\text{Spin}}) \hookrightarrow V_{\text{Spin}} \otimes V_{\text{Spin}} \xrightarrow{\delta \otimes \text{id}_{V_{\text{Spin}}}} V_{\text{Spin}}^* \otimes V_{\text{Spin}} = \text{End}(V_{\text{Spin}}).$$

Then there are two subtly different cases to distinguish.

Case 1: If m is odd then we construct π as follows. Applying the constructions from Section 3.1 we have an isomorphism of representations (6),

$$\kappa_-^{-1} : \text{End}(V_{\text{Spin}}) \rightarrow \text{Cl}^-(V) \rightarrow \bigoplus_{k=0}^m \bigwedge^{2k+1} V.$$

Because m is odd we have a projection onto the summand with $k = \frac{m-1}{2}$,

$$\text{pr}_{\bigwedge^m} : \bigoplus_{k=0}^m \bigwedge^{2k+1} V \rightarrow \bigwedge^m V.$$

By contracting with $(-1)^{\frac{m(m+1)}{2}} v_1^* \wedge \cdots \wedge v_{2m+1}^*$ we get an equivariant isomorphism

$$c : \bigwedge^m V \rightarrow \bigwedge^{m+1} V^*.$$

Finally, we have an equivariant isomorphism

$$d : \bigwedge^{m+1} V^* \rightarrow \bigwedge^{m+1} V$$

defined using the isomorphism $V \cong V^*$ given by the quadratic form and made explicit in the end of Subsection 3.1. Composing $\iota_{V_{\text{Spin}}}$ with these four maps gives us our homomorphism of representations

$$\pi : \text{Sym}^2(V_{\text{Spin}}) \longrightarrow \bigwedge^{m+1} V.$$

Case 2: Suppose m is even. In this case we use the even part of the Clifford algebra of V , namely we use the inverse of the isomorphism from (5)

$$\kappa_+^{-1} : \text{End}(V_{\text{Spin}}) \rightarrow \text{Cl}^+(V) \rightarrow \bigoplus_{k=0}^m \bigwedge^{2k} V.$$

Since m is even we have a projection onto the middle summand, $k = \frac{m}{2}$,

$$\text{pr}_{\bigwedge^m} : \bigoplus_{k=0}^m \bigwedge^{2k} V^* \rightarrow \bigwedge^m V.$$

Finally we use the isomorphism of representations c as in Case 1,

$$c : \bigwedge^m V \xrightarrow{\sim} \bigwedge^{m+1} V^*$$

as well as the map

$$d : \bigwedge^{m+1} V^* \rightarrow \bigwedge^{m+1} V.$$

Composing $\iota_{V_{\text{Spin}}}$ with these four maps gives us our homomorphism of representations

$$\pi : \text{Sym}^2(V_{\text{Spin}}) \longrightarrow \bigwedge^{m+1} V$$

in the case where m is even.

3.4.2 Statement

Definition 3.2. Corresponding to the quadratic denominators in W_t we define elements of $\text{Sym}^2(V_{\text{Spin}})$ by

$$\mathcal{D}_{(j)} := \sum_I (-1)^{s(I)} w_{\rho_{m+1-j}^I} w_{\mu_{m+1-j}^I}$$

and

$$\mathcal{N}_{(j)} := \sum_I (-1)^{s(I)} w_{\rho_{m+1-j,+}^I} w_{\mu_{m+1-j,+}^I}$$

where the sums are over all subsets $I \subset \{1, \dots, m+1-j\}$ and $j = 2, \dots, m$.

We will prove that for all $j = 2, \dots, m$:

Proposition 3.6.

$$\sum_I (-1)^{s(I)} p_{\rho_{m+1-j}^I}(\bar{u}_2) p_{\mu_{m+1-j}^I}(\bar{u}_2) = \Delta_{m+2-j, \dots, 2m+2-j}^{m+1, \dots, 2m+1}(\bar{u}_2)$$

and

$$\sum_I (-1)^{s(I)} p_{\rho_{m+1-j,+}^I}(\bar{u}_2) p_{\mu_{m+1-j,+}^I}(\bar{u}_2) = \Delta_{m+1-j, m+3-j, \dots, 2m+2-j}^{m+1, \dots, 2m+1}(\bar{u}_2)$$

where the sums are over all subsets $I \subset \{1, \dots, m+1-j\}$.

Remark. Note that this proposition gives us an alternative definition of \check{X}° in terms of non-vanishing of minors.

3.4.3 Proof

To prove Proposition 3.6, we will need to compare $\mathcal{D}_{(j)}, \mathcal{N}_{(j)} \in \text{Sym}^2(V_{\text{Spin}})$ to the elements of $\bigwedge^{m+1} V$ defined below.

Definition 3.3. Inside the exterior power $\bigwedge^{m+1} V$, if $2 \leq j \leq m$ we consider the elements

$$\begin{aligned} v_{(j)}^\wedge &:= v_j \wedge \cdots \wedge v_{j+m} \\ v_{(j),+}^\wedge &:= v_{j-1} \wedge v_{j+1} \wedge \cdots \wedge v_{j+m} \end{aligned}$$

of $\bigwedge^{m+1} V$.

We will show :

Proposition 3.7. *The projection map $\pi : \text{Sym}^2(V_{\text{Spin}}) \longrightarrow \bigwedge^{m+1} V$ takes $\mathcal{D}_{(j)}$ to $v_{(j)}^\wedge$ and $\mathcal{N}_{(j)}$ to $v_{(j),+}^\wedge$.*

We will in fact prove this proposition only for the denominators $\mathcal{D}_{(j)}$, the case of the numerators $\mathcal{N}_{(j)}$ being extremely similar.

Definition 3.4. If $I = \{1 \leq i_1 < \cdots < i_r \leq 2m+1\}$, we define v_I to be the product $v_{i_1} \cdots v_{i_r}$ in $\text{Cl}(V)$. For $I = \{j, j+1, \dots, j+m\}$ we also denote v_I by $v_{(j)}$, so $v_{(j)} = v_j v_{j+1} \cdots v_{j+m}$. Moreover, if L is a subset of $\{j, \dots, m\}$, we write $v_{(j)}^L$ for the Clifford algebra element obtained from the product $v_{(j)}$ by removing all of the factors v_l and $\bar{v}_l = v_{2m+2-l}$ for which $l \in L$.

Lemma 3.8. *The map $\iota_{V_{\text{Spin}}} : \text{Sym}^2(V_{\text{Spin}}) \hookrightarrow \text{End}(V_{\text{Spin}})$ maps $\mathcal{D}_{(j)}$ to*

$$\beta_{m,j} \cdot \sum_I \left[w_{\mu_{j-1}^I}^* \otimes w_{\mu_{m+1-j}^I} + (-1)^{(m+1-j)(j-1)} w_{\rho_{j-1}^I}^* \otimes w_{\rho_{m+1-j}^I} \right] \quad (14)$$

where

$$\beta_{m,j} := \frac{(-1)^{\frac{(m+1-j)(m+2-j)}{2}}}{2}$$

and the sum is over all subsets I of $\{1, \dots, m+1-j\}$.

Proof. First $w_{\rho_{m+1-j}^I} w_{\mu_{m+1-j}^I}$ maps to

$$\frac{1}{2} (w_{\rho_{m+1-j}^I} \otimes w_{\mu_{m+1-j}^I} + w_{\mu_{m+1-j}^I} \otimes w_{\rho_{m+1-j}^I}) \in V_{\text{Spin}} \otimes V_{\text{Spin}}.$$

Then according to Lemma 3.5 :

$$\begin{aligned} w_{\rho_{m+1-j}^I} &\mapsto (-1)^{\frac{(m+1-j)(m+2-j)}{2} - s(I)} w_{\mu_{j-1}^I}^* \in V_{\text{Spin}}^* \\ w_{\mu_{m+1-j}^I} &\mapsto (-1)^{\frac{m(m+1)}{2} - \frac{j(j-1)}{2} + s(I)} w_{\rho_{j-1}^I}^* \in V_{\text{Spin}}^*, \end{aligned}$$

hence the result. \square

We now need to map the element (14) to the Clifford algebra of V .

Proposition 3.9.

$$\mathcal{D}_{(j)} \mapsto \frac{(-1)^{\frac{m(m+1)}{2}}}{2} [2v_{1,\dots,m+1-j,2m+3-j,\dots,2m+1} + \sum_{I \subsetneq \{1,\dots,m+1-j\}} \left(\prod_{l \in \{1,\dots,m+1-j\} \setminus I} (-1)^l \right) v_{I \cup \{2m+3-j,\dots,m+j\} \cup \bar{I}}] \in \text{Cl}(V). \quad (15)$$

Proof. We assume $j > \frac{m+1}{2}$, the other case being symmetric. For convenience, let us denote the right-hand side as $A_{(j)} \in \text{Cl}(V)$. Because of the definition of the Clifford algebra :

$$v_{1,\dots,m+1-j,2m+3-j,\dots,2m+1} = (-1)^{m(m+1-j)} v_{\{1,\dots,m+1-j\} \cup \overline{\{1,\dots,m+1-j\}}} t_{(j)},$$

where $t_{(j)} = v_{2m+3-j} \dots v_{m+j}$. Similarly

$$v_{I \cup \{2m+3-j,\dots,m+j\} \cup \bar{I}} = (-1)^{m|I|} v_{I \cup \bar{I}} t_{(j)}.$$

We will use two lemmas :

Lemma 3.10. *Let I be a subset of $\{1, \dots, m\}$. Then*

$$v_{I \cup \bar{I}} \mapsto \left(\prod_{i \in I} \epsilon(i) \right) \sum_L w_L^* \otimes w_L \in \text{End}(V_{\text{Spin}}),$$

where the sum is over all subsets L of $\{1, \dots, m\}$ containing I .

Proof of lemma 3.10. First notice that

$$v_i^- \cdot w_L = \begin{cases} 0 & \text{if } i \notin L \\ (-1)^{\#\{l \in L \mid l < i\}} \epsilon(i) w_{L \setminus \{i\}} & \text{otherwise,} \end{cases}$$

and

$$v_i v_i^- \cdot w_L = \begin{cases} 0 & \text{if } i \notin L \\ \epsilon(i) w_L & \text{otherwise.} \end{cases}$$

Hence $v_{I \cup \bar{I}}$ is zero unless $L \supset I$. Now assume $L \supset I$ and write $I = \{i_1 < i_2 < \dots < i_r\}$. From the definition of the Clifford algebra, it follows that $v_{I \cup \bar{I}} = \prod_{p=1}^r v_{i_p} v_{i_p}^-$. Hence :

$$v_{I \cup \bar{I}} \cdot w_L = \left(\prod_{p=1}^r \epsilon(i_p) \right) w_L.$$

The claim follows. \square

Lemma 3.11. *The element $t_{(j)} = v_{2m+3-j} \dots v_{m+j}$ of $\text{Cl}(V)$ maps to*

$$\left(\prod_{p=m+2-j}^{j-1} \epsilon(p) \right) \sum_{K_1, K_2} (-1)^{m|K_1|} w_{K_1 \cup \{m+2-j, \dots, j-1\} \cup K_2}^* \otimes w_{K_1 \cup K_2} \in \text{End}(V_{\text{Spin}}),$$

where K_1 is any subset of $\{1, \dots, m+1-j\}$ and K_2 is any subset of $\{j, \dots, m\}$.

Proof of lemma 3.11. As in the proof of Lemma 3.10, we notice that $t_{(j)} \cdot w_L = 0$ if $L \not\supset \{m+2-j, \dots, j-1\}$. Now write $L = L_1 \cup \{m+2-j, \dots, j-1\} \cup L_2$, where $L_1 \subset \{1, \dots, m+1-j\}$ and $L_2 \subset \{j, \dots, m\}$. We have

$$v_{m+j} \cdot w_L = (-1)^{m|L_1|} \epsilon(m+2-j) w_{L_1 \cup \{m+3-j, \dots, j-1\} \cup L_2}.$$

Recursively, we obtain :

$$t_{(j)} \cdot w_L = (-1)^{m|L_1|} \left(\prod_{p=m+2-j}^{j-1} \epsilon(p) \right) w_{L_1 \cup L_2},$$

hence the lemma. \square

Now to prove Proposition 3.9, first assume $L = \{1, \dots, j-1\} \cup L_2$, where $L_2 \subset \{j, \dots, m\}$. Then

$$v_{1, \dots, m+1-j, 2m+3-j, \dots, 2m+1} \cdot w_L = \left(\prod_{p=1}^{j-1} \epsilon(p) \right) w_{L_1 \cup L_2},$$

and

$$\left(\prod_{l \in \{1, \dots, m+1-j\} \setminus I} (-1)^l \right) v_{I \cup \{2m+3-j, \dots, m+j\} \cup \bar{I}} \cdot w_L$$

is equal to

$$\left(\prod_{p=1}^{j-1} \epsilon(p) \right) (-1)^{|I|} (-1)^{m+1-j} w_{L_1 \cup L_2}.$$

Hence

$$\begin{aligned} A_{(j)} \cdot w_L &= \left(\prod_{p=1}^{j-1} \epsilon(p) \right) \left[2 + (-1)^{m+1-j} \sum_{I \subsetneq \{1, \dots, m+1-j\}} (-1)^{|I|} \right] w_{L_1 \cup L_2} \\ &= \left(\prod_{p=1}^{j-1} \epsilon(p) \right) w_{L_1 \cup L_2}. \end{aligned}$$

Now assume $L = L_1 \cup \{m+2-j, \dots, j-1\} \cup L_2$, where $L_1 \subsetneq \{1, \dots, m+1-j\}$ and $L_2 \subset \{j, \dots, m\}$. Then

$$A_{(j)} \cdot w_L = \left(\prod_{p=1}^{j-1} \epsilon(p) \right) (-1)^{m|L_1|} (-1)^{(m+1)(m+1-j)} \sum_{I \subset L_1} (-1)^{|I|} w_{L_1 \cup L_2}.$$

Finally :

$$A_{(j)} \cdot w_L = \begin{cases} 0 & \text{if } L_1 \neq \emptyset \\ \left(\prod_{p=1}^{j-1} \epsilon(p) \right) (-1)^{(m+1)(m+1-j)} w_{L_2} & \text{otherwise.} \end{cases}$$

Looking precisely at the expression of $\mathcal{D}_{(j)}$ in $\text{End}(V_{\text{Spin}})$, this concludes the proof of the proposition. \square

Corollary 3.12. *We have :*

$$\text{pr}_{\wedge^m} \circ \kappa_{\pm}^{-1} \circ \iota_{V_{\text{Spin}}}(\mathcal{D}_{(j)}) = (-1)^{\frac{m(m+1)}{2}} v_1 \wedge \dots \wedge v_{m+1-j} \wedge v_{2m+3-j} \wedge \dots \wedge v_{2m+1}$$

where κ_{\pm} is κ_- if m is odd and κ_+ otherwise.

Proof. The result is a simple consequence of Proposition 3.9 and of the definition of the antisymmetrisation maps (3) and (4). \square

We can now prove Proposition 3.7 :

Proof of Proposition 3.7. From Corollary 3.12, we know that $\mathcal{D}_{(j)}$ maps to $(-1)^{\frac{m(m+1)}{2}} v_1 \wedge \dots \wedge v_{m+1-j} \wedge v_{2m+3-j} \wedge \dots \wedge v_{2m+1}$ in $\wedge^m V$. Now the latter element is mapped by the contraction c to

$$(-1)^{(m+1)(j-1)} v_{m+2-j}^* \wedge \dots \wedge v_{2m+2-j}^*.$$

Then we map this to $\wedge^{m+1} V$ using the isomorphism d . We have

$$\begin{aligned} v_{m+2-j}^* \wedge \dots \wedge v_{2m+2-j}^* &\mapsto \left(\prod_{i=m+2-j}^m \epsilon(i) \right) \left(\prod_{k=j}^m \epsilon(k) \right) v_{j+m} \wedge v_{j+m+1} \wedge \dots \wedge v_j \\ &\mapsto (-1)^{j^2+m^2-mj+1} v_{(j)}^{\wedge}. \end{aligned}$$

Now

$$\mathcal{D}_{(j)} \mapsto v_{(j)}^{\wedge},$$

which concludes the proof. \square

4 Some relations in the quantum cohomology

In [Rie08], the second author proved an isomorphism between the quantum cohomology ring of $X = G^\vee/P^\vee$ and the Jacobi ring of the LG-model $(\mathcal{R}, \mathcal{F}_h)$ (either at fixed quantum parameter $q = e^h$ as in Corollary 2.4 or over the ring $\mathbb{C}[q, q^{-1}]$). By Theorem 3.1 together with Claim 1 our LG-model (\check{X}, W_t) should be isomorphic to this one, and therefore related to the quantum cohomology ring of $LG(m)$ in the same way. Therefore we expect the denominators appearing in the expression of W_t , once written with Schubert classes replacing the Plücker coordinates, to represent invertible elements in this quantum cohomology ring. We have a precise conjecture for which elements these are.

Conjecture 4.1. *The following relation holds in the quantum cohomology of $LG(m)$ for all $1 \leq l \leq m-1$:*

$$\sum_{J \subset \{1, \dots, l\}} (-1)^{s(J)} \sigma_{\rho_l^J} \star \sigma_{\mu_l^J} = q^l. \quad (16)$$

Remark. If $l = 1$, the relation (16) is a consequence of the quantum Chevalley formula 2.2. Indeed, this formula implies that

$$\sigma_1 \star \sigma_m = \sigma_{m,1} + q,$$

which, rewritten as

$$\sigma_1 \star \sigma_m - \sigma_\emptyset \star \sigma_{m,1} = q,$$

is exactly the relation (16) with $l = 1$. For $l > 1$ however, to the best of the authors' knowledge, the relations (16) are new.

5 The B-model connection

In this section we briefly state an explicit mirror symmetry conjecture for our superpotential W_t . Namely the conjecture asserts that a Gauss-Manin connection associated to W should recover connections defined on the A -model side by Dubrovin and Givental, see [Dub96, Giv96, CK99].

Let $X = LG(m)$. Consider $H^*(X, \mathbb{C}[\hbar, e^t])$ as space of sections on a trivial bundle with fiber $H^*(X)$ and let

$${}^A\nabla_{\partial_t} S := \frac{dS}{dt} - \frac{1}{\hbar} \sigma^\square \star_{e^t} S \quad (17)$$

$${}^A\nabla_{\hbar\partial_\hbar} S := \hbar \frac{\partial S}{\partial \hbar} + \frac{1}{\hbar} c_1(TX) \star_{e^t} S \quad (18)$$

define a meromorphic flat connection on this bundle.¹ This is our A -model side.

¹ We are using the convenient notation e^t for q and ∂_t for $q\partial_q$. Also, for simplicity of the statement of the conjecture, we have removed the grading operator contained in Dubrovin's original definition of ${}^A\nabla_{\hbar\partial_\hbar}$.

For the B -model let $N = \frac{m(m+1)}{2}$ denote the dimension of \check{X} . Recall that \check{X}° is $\text{OG}^{\text{co}}(m+1, 2m+1)$ with an anti-canonical divisor removed. Therefore there is an up to scalar unique non-vanishing holomorphic N -form on \check{X}° which we will fix and call ω . Let $\Omega^k(\check{X}^\circ)$ denote the space of all holomorphic k -forms.

Definition 5.1. Define the $\mathbb{C}[\hbar, e^t]$ -module

$$G_0^{W_t} := \Omega^N(X)[\hbar, e^t] / (\hbar d + dW_t \wedge -) \Omega^{N-1}(X)[\hbar, e^t].$$

It has a meromorphic (Gauss-Manin) connection given by

$${}^B\nabla_{\partial_t}[\alpha] = \frac{\partial}{\partial t}[\alpha] - \frac{1}{\hbar} \left[\frac{\partial W_t}{\partial t} \alpha \right], \quad (19)$$

$${}^B\nabla_{\partial_\hbar}[\alpha] = \frac{\partial}{\partial \hbar}[\alpha] + \frac{1}{\hbar^2} [W_t \alpha]. \quad (20)$$

We conjecture that the function W_t is cohomologically tame [Sab99] and the elements $[p_\lambda \omega]$ freely generate $G_0^{W_t}$, where the p_λ 's are the Plücker coordinate on $\text{OG}^{\text{co}}(m+1, V^*)$ and λ runs through the strict partitions inside an $m \times m$ box.

Independently of this we conjecture the following.

Conjecture 5.1. *The differential operators $\hbar {}^B\nabla_{\partial_t}$ and $\hbar {}^B\nabla_{\hbar \partial_\hbar}$ preserve the $\mathbb{C}[\hbar, e^t]$ -submodule $\bar{G}_0^{W_t}$ of $G_0^{W_t}$ generated by the $[p_\lambda \omega]$. Moreover the assignment $\sigma^\lambda \mapsto [p_\lambda \omega]$ defines an isomorphism of $H^*(X, \mathbb{C}[\hbar, e^t])$ with $\bar{G}_0^{W_t}$ under which ${}^A\nabla$ is identified with ${}^B\nabla$.*

A Appendix

We may give also a Laurent polynomial expression for W_t restricted to a particular torus. Namely let us pull back W_t to the open subset of \check{X} defined as the image of the map $(\mathbb{C}^*)^N \hookrightarrow P \backslash G$ which sends (b_1, \dots, b_N) to $P\bar{u}_2$, where as in Section 3.3

$$\bar{u}_2 = y_m(b_N) \dots y_2(b_{N-m+2}) y_1(b_{N-m+1}) \dots y_m(b_3) y_{m-1}(b_2) y_m(b_1). \quad (21)$$

Proposition A.1 (Laurent polynomial restriction of W_t). *The Landau-Ginzburg model W_t of $X = \text{LG}(m)$ defined in Theorem 2.4 restricts to the open torus defined above to give*

$$\tilde{W}_t(b_1, \dots, b_N) = \sum_{j=1}^N b_j + e^t \frac{\mathcal{N}(b_1, \dots, b_N)}{\prod_{j=1}^N b_j},$$

where

$$\mathcal{N}(b_1, \dots, b_N) := \sum b_{j_{i_1}} \dots b_{j_{i_{N-m}}},$$

and the sum is over all subsets $\{i_1 < \dots < i_{N-m}\}$ of $\{1, \dots, N\}$ such that $(s_{j_{i_1}} \dots s_{j_{i_N}}) s_1 \dots s_m$ is a reduced expression for w^P .

Proof. We will rename the coordinates b_i when convenient by $a_{i,j}$, in terms of which \bar{u}_2 is given by

$$(y_m(a_{m,m})y_{m-1}(a_{m-1,m})\dots y_1(a_{1,m}))\dots(y_m(a_{m,2})y_{m-1}(a_{m-1,2}))y_m(a_{m,1}).$$

As a consequence of the shape of \bar{u}_2 and the definition of the y_i , we immediately obtain :

$$f_i^*(\bar{u}_2) = \sum_{j=m+1-i}^m a_j^{(i)}. \quad (22)$$

We now need to compute the $e_i^*(u_1)$, where u_1 is such that $u_1 e^h \dot{w}_P \bar{u}_2 \in B_- \dot{w}_0$.

Lemma A.2.

$$e_i^*(u_1) = 0 \text{ for all } 1 \leq i \leq m-1 \quad (23)$$

Proof of Lemma A.2. From [Rie08], we know that

$$\begin{aligned} e_i^*(u_1) &= \frac{\langle u_1^{-1} v_{\omega_i}^-, e_i \cdot v_{\omega_i}^- \rangle}{\langle u_1^{-1} v_{\omega_i}^-, v_{\omega_i}^- \rangle} \\ &= \frac{\langle e^h \dot{w}_P \bar{u}_2 \dot{w}_0^{-1} v_{\omega_i}^-, e_i \cdot v_{\omega_i}^- \rangle}{\langle e^h \dot{w}_P \bar{u}_2 \dot{w}_0^{-1} v_{\omega_i}^-, v_{\omega_i}^- \rangle} \\ &= \frac{\langle e^h \dot{w}_P \bar{u}_2 v_{\omega_i}^+, e_i \cdot v_{\omega_i}^- \rangle}{\langle e^h \dot{w}_P \bar{u}_2 v_{\omega_i}^+, v_{\omega_i}^- \rangle}. \end{aligned}$$

Now $e_i^*(u_1) = 0$ if and only if $\langle \bar{u}_2 v_{\omega_i}^+, \dot{w}_P^{-1} e_i \cdot v_{\omega_i}^- \rangle = 0$. The vector $\dot{w}_P^{-1} e_i \cdot v_{\omega_i}^-$ is in the μ -weight space of the i -th fundamental representation, where $\mu := w_P^{-1} s_i(-\omega_i)$. Moreover, $\bar{u}_2 \in B_+(\dot{w}^P)^{-1} B_+$, hence it can only have non-zero components down to the weight space of weight $(w^P)^{-1}(\omega_i) = w_P^{-1}(-\omega_i)$. However, μ is lower than $w_P^{-1}(-\omega_i)$ when $i \neq m$. \square

We are left with computing $e_m^*(u_1)$:

Lemma A.3.

$$e_m^*(u_1) = e^t \frac{\mathcal{N}(b_1, \dots, b_N)}{\prod_{j=1}^N b_j} \quad (24)$$

Proof of Lemma A.3. As in the proof of Lemma A.2, we have

$$\begin{aligned} e_m^*(u_1) &= \frac{\langle e^h \dot{w}_P \bar{u}_2 v_{\omega_m}^+, e_m \cdot v_{\omega_m}^- \rangle}{\langle e^h \dot{w}_P \bar{u}_2 v_{\omega_m}^+, v_{\omega_m}^- \rangle} \\ &= (\omega_m + \alpha_m - \omega_m)(e^h) \frac{\langle \dot{w}_P \bar{u}_2 v_{\omega_m}^+, e_m \cdot v_{\omega_m}^- \rangle}{\langle \dot{w}_P \bar{u}_2 v_{\omega_m}^+, v_{\omega_m}^- \rangle} \\ &= e^t \frac{\langle \dot{w}_P \bar{u}_2 v_{\omega_m}^+, e_m \cdot v_{\omega_m}^- \rangle}{\langle \dot{w}_P \bar{u}_2 v_{\omega_m}^+, v_{\omega_m}^- \rangle}. \end{aligned}$$

Indeed, $\alpha_m(e^h) = e^t$. Moreover, $\langle w_P \bar{u}_2 v_{\omega_m}^+, v_{\omega_m}^- \rangle = \langle \bar{u}_2 v_{\omega_m}^+, w_P^{-1} v_{\omega_m}^- \rangle = \langle \bar{u}_2 v_{\omega_m}^+, v_{\omega_m}^- \rangle$. Now the only way to go from the lowest weight vector $v_{\omega_m}^-$ of the m -th fundamental representation to the highest $v_{\omega_m}^+$ is to apply w_0 . Since $\bar{u}_2 \in B(w^P)^{-1}B$, it follows that we need to take all factors of \bar{u}_2 , hence $\langle w_P \bar{u}_2 v_{\omega_m}^+, v_{\omega_m}^- \rangle = \prod_{j=1}^N b_j$.

Now we prove that $\langle w_P \bar{u}_2 v_{\omega_m}^+, e_m \cdot v_{\omega_m}^- \rangle = \mathcal{N}(b_1, \dots, b_N)$. Indeed :

$$\langle w_P \bar{u}_2 v_{\omega_m}^+, e_m \cdot v_{\omega_m}^- \rangle = \langle \bar{u}_2 v_{\omega_m}^+, w_P^{-1} e_m \cdot v_{\omega_m}^- \rangle,$$

and the weight of the vector $w_P^{-1} e_m \cdot v_{\omega_m}^-$ is $\mu' := \frac{1}{2}(\epsilon_1 - \epsilon_2 - \dots - \epsilon_m)$. Now consider the Weyl group element

$$w' := s_m(s_{m-1}s_m) \dots (s_2 \dots s_{m-1}s_m).$$

We have

$$w' \cdot \omega_m = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \dots - \epsilon_m).$$

Hence the way to the μ' -weight space is through one of the reduced expression for w' , which concludes the proof of the claim. \square

Now the proof of Proposition A.1 follows immediately from Theorem 2.4 and the equations (22), (23) and (24). \square

The expression for the Landau-Ginzburg model in Proposition A.1 is quite close to the usual expression for the Landau-Ginzburg model of projective space \mathbb{P}^n , which looks like :

$$W_t^{\mathbb{P}^n} = x_1 + x_2 + \dots + x_n + \frac{e^t}{x_1 x_2 \dots x_n}.$$

Indeed, It is the sum of as many parameters as the dimension of the variety, plus a more complicated e^t -term depending on those parameters. To the best of our knowledge, this expression is new for $\text{LG}(m)$ with $m > 2$. However, for the three-dimensional quadric $\text{LG}(2)$, we obtain :

$$W_t^{\text{LG}(2)} = a_{2,1} + a_{1,2} + a_{2,2} + e^t \frac{a_{2,1} + a_{2,2}}{a_{2,1} a_{1,2} a_{2,2}},$$

which, up to a toric change of coordinates, corresponds to one of the expressions of [Prz07].

References

- [BCFKvS00] Victor V. Batyrev, Ionuț Ciocan-Fontanine, Bumsig Kim, and Duco van Straten. Mirror symmetry and toric degenerations of partial flag manifolds. *Acta Math.*, 184(1):1–39, 2000.

- [CK99] David Cox and Sheldon Katz. *Mirror Symmetry and Algebraic Geometry*. American Mathematical Soc., 1999.
- [Dub96] Boris Dubrovin. Geometry of 2d topological field theories. *Integrable Systems and Quantum Groups*, 1620:120 – 348, 1996.
- [FW04] William Fulton and Christopher Woodward. On the quantum product of Schubert classes. *J. Algebraic Geom.*, 13(4):641–661, 2004.
- [Gin97] Ginzburg, V. Perverse sheaves on a loop group and Langlands duality. preprint, 1997.
- [Giv96] Alexander B. Givental. Equivariant Gromov-Witten invariants. *IMRN*, 13:613–663, 1996.
- [GK95] Alexander Givental and Bumsig Kim. Quantum cohomology of flag manifolds and Toda lattices. *Comm. Math. Phys.*, 168(3):609–641, 1995.
- [Kim99] Bumsig Kim. Quantum cohomology of flag manifolds g/b and quantum toda lattices. *Annals of Mathematics, Second Series*, 149(No. 1):129–148, Jan., 1999.
- [Lus83] G. Lusztig. Singularities, character formulas and a q -analog of weight multiplicities. *Astérisque*, 101-102:208–229, 1983.
- [MR12] R. Marsh and K. Rietsch. The B -model connection and T -equivariant mirror symmetry for Grassmannians. 2012.
- [MV07] I. Mirković and K. Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Ann. of Math. (2)*, 166(1):95–143, 2007.
- [Pet97] D. Peterson. Quantum cohomology of G/P . Lecture Course, MIT, Spring Term, 1997.
- [Prz07] Victor Przyjalkowski. On Landau-Ginzburg models for Fano varieties. *Commun. Number Theory Phys.*, 1(4):713–728, 2007.
- [Rie08] Konstanze Rietsch. A mirror symmetric construction of $qH_T^*(G/P)_{(q)}$. *Adv. Math.*, 217(6):2401–2442, 2008.
- [Sab99] Claude Sabbah. Hypergeometric period for a tame polynomial. *C. R. Acad. Sci. Paris Ser. I Math.*, 328(7):603–608, 1999.
- [Var04] V. S. Varadarajan. *Supersymmetry for mathematicians: an introduction*, volume 11 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2004.